

# Uniform Invariance Principle and Synchronization. Robustness with Respect to Parameter Variation<sup>1</sup>

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The objective of this work is to obtain uniform estimates, with respect to parameters, of the attractor and of the basin of attraction of a dynamical system and to apply these results to analyze the roughness of the synchronization of two subsystems. These estimates are obtained through a uniform version of the invariance principle of La Salle which is stated and proved in this work. © 2001

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*Key Words:* invariance principle; synchronization.

## 1. INTRODUCTION

The invariance principle has been one of the most important tools used to study the asymptotic behavior of solutions of differential equations. It was first stated and proved for autonomous differential equations defined on finite dimensional spaces [13, 14] by J. P. LaSalle and it was successfully extended to differential equations defined on infinite dimensional spaces, see Hale [12], and Slemrod [22], including to functional differential equations, see Hale and Lunel [9]. It was also extended to non-autonomous differential equations, see LaSalle [15] for the periodic case,

<sup>1</sup> Partially supported by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo), CAPES (Função Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) and CNPq (Conselho Nacional de Desenvolvimento Tecnológico), Brazil.



Miller [17] for the almost periodic case and Sell [21] for more general ordinary differential equations, and also to nonautonomous retarded equations, see Rodrigues [19]. LaSalle [16] also obtained an extension for difference equations.

Although, most applications of the invariance principle are concerned with convergence to equilibrium, in this paper it is shown that it can also be used to study synchronization between solutions of coupled differential equations.

Synchronization is an important concept that has been extensively used by researchers of applied sciences such as electrical and mechanical engineering, biology, and physics. It has also successfully been used on communication systems for codification of information; see Cuomo and Hoppenheim [7], Yang and Chua [24], and Peccora *et al.* [18].

Mathematical methods to study synchronization between chaotic systems were presented in Fujisaka and Yamada [8], in Afraimovich *et al.* [2], and in Wu and Chua [23]. Abstract results and the robustness with respect to parameters variation and uniform dissipativeness were obtained in Rodrigues [20] and Afraimovich and Rodrigues [1].

For infinite dimensional systems some results are presented in Rodrigues [20], Carvalho *et al.* [4], Hale [10, 11] and Afraimovich *et al.* [3].

The object of this paper is to present a more general version of the invariance principle in which the derivative of the Liapunov function is not required to always be negative semidefinite and the parameters are allowed to vary on a certain range. In many complex engineering systems and systems whose solutions present a complicated or chaotic behavior, it may not be easy to find a Liapunov function such that its derivative along the solutions is not negative. Therefore, the results presented here will be helpful in such cases. It is important to point out that, in this paper, the expression "Liapunov function" should be understood in a wider sense in which its derivative along the solutions may also be positive.

The uniform invariance principle proposed in this paper is useful to obtain concrete upper bounds for attractors and for the attraction basin and also to study synchronization. Estimates in some examples, such as Lorenz equation and power systems are obtained.

This paper is organized as follow. In Section 2, the theoretical results and some examples including good estimates for a single Lorenz system are presented. In Section 3, estimates of the attractor and studies of synchronization for coupled Lorenz systems, power systems, etc. are evaluated. We emphasize that, for the models of power systems, we proved synchronization even when the the decoupled individual subsystems are conservative. The coupling parameters are estimated in order to accomplish the synchronization. Concluding remarks are presented in Section 4.

## 2. THE UNIFORM INVARIANCE PRINCIPLE

This section starts with a review of the usual invariance principle. Consider the following autonomous differential equation:

$$\dot{x} = f(x). \quad (2.1)$$

**THEOREM 2.1.** *Let  $V: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be  $C^1$  functions. Let  $L > 0$  be a constant such that  $\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}$  is bounded. Suppose that  $\dot{V}(x) \leq 0$  for every  $x \in \Omega_L$  and define  $E := \{x \in \Omega_L : \dot{V}(x) = 0\}$ . Let  $B$  be the largest invariant set contained in  $E$ . Then every solution of (2.1) starting in  $\Omega_L$  converges to  $B$  as  $t \rightarrow \infty$ .*

A global version of this theorem can be stated as follows:

**THEOREM 2.2.** *Let  $V: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be  $C^1$  functions. Suppose that  $\dot{V}(x) \leq 0$  for every  $x \in \mathbb{R}^n$  and define  $E := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ . Let  $B$  be the largest invariant set contained in  $E$ . Then every solution of (2.1) which is bounded for  $t \geq 0$  converges to  $B$  as  $t \rightarrow \infty$ .*

For  $\lambda \in A \subset \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$ , consider the following autonomous differential equation:

$$\dot{x} = f(x, \lambda). \quad (2.2)$$

**THEOREM 2.3. (The Uniform Invariance Principle).** *Suppose  $f: \mathbb{R}^n \times A \rightarrow \mathbb{R}^n$  and  $V: \mathbb{R}^n \times A \rightarrow \mathbb{R}$  are  $C^1$  functions and  $a, b, c: \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous functions. Assume that for any  $(x, \lambda) \in \mathbb{R}^n \times A$ , one has:*

$$a(x) \leq V(x, \lambda) \leq b(x), \quad -\dot{V}(x, \lambda) \geq c(x).$$

*For  $\rho > 0$  let  $\mathcal{A}_\rho := \{x \in \mathbb{R}^n : a(x) < \rho\}$  (see Fig. 1). Assume that  $\mathcal{A}_\rho$  is non-empty and bounded.*

*Consider the sets*

$$\mathcal{B}_\rho := \{x \in \mathbb{R}^n : b(x) < \rho\}, \quad C := \{x \in \mathbb{R}^n : c(x) < 0\}$$

$$E_\rho := \{x \in \mathcal{A}_\rho : c(x) = 0\}.$$

*Suppose now that  $\sup_{x \in C} b(x) \leq R < \rho$  and define the sets*

$$A_R := \{x \in \mathbb{R}^n : a(x) \leq R\} \quad \text{and} \quad B_R := \{x \in \mathbb{R}^n : b(x) \leq R\}.$$

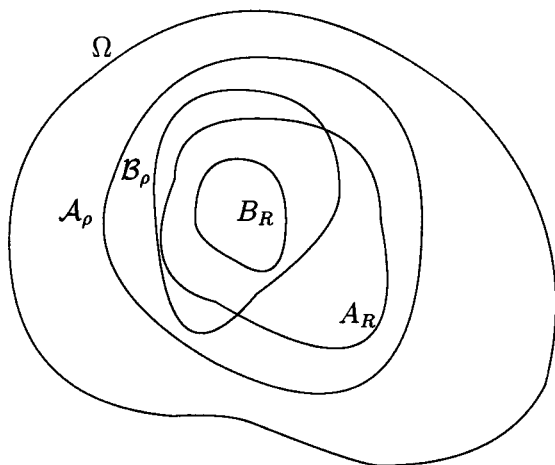


FIG. 1. Illustration of the sets  $\mathcal{A}_\rho$ ,  $\mathcal{B}_\rho$ ,  $A_R$ , and  $B_R$ .

If  $\lambda$  is a fixed parameter in  $\Lambda$  and all the previous conditions are satisfied then for  $x_0 \in \mathcal{B}_\rho$  the solution  $\varphi(t, x_0, \lambda)$  is defined in  $[0, \infty)$  and the following holds:

(I) if  $x_0 \in B_R$  then  $\varphi(t, x_0, \lambda) \in A_R$ , for  $t \geq 0$  and  $\varphi(t, x_0, \lambda)$  tends to the largest invariant set of (2.2) contained in  $A_R$ , as  $t \rightarrow \infty$ .

(II) if  $x_0 \in \mathcal{B}_\rho - B_R$  then  $\varphi(t, x_0, \lambda)$  tends to the largest invariant set of (2.2) contained in  $A_R \cup E_\rho$ .

*Proof.* First of all it is important to point out that  $\mathcal{B}_\rho \subset \{x \in \mathbb{R}^n : V(x, \lambda) \leq \rho\} \subset \mathcal{A}_\rho$  and  $-\dot{V}(x_0, \lambda) \geq c(x_0) \geq 0$ , for each  $\lambda \in \Lambda$  and each  $x_0 \in \mathbb{R}^n - B_R$ .

We consider two cases: (I) For  $x_0 \in B_R$ , let  $[0, t_+)$  be the maximum interval of existence of the solution  $\varphi(t, x_0, \lambda)$ , of (2.2). Suppose there exists  $\tilde{t} \in [0, t_+)$  such that  $\varphi(\tilde{t}, x_0, \lambda) \notin A_R$ . Then  $a(\varphi(\tilde{t}, x_0, \lambda)) > R$ . Then  $V(x_0, \lambda) = V(\varphi(0, x_0, \lambda), \lambda) \leq b(\varphi(0, x_0, \lambda)) = b(x_0) \leq R$  and  $V(\varphi(\tilde{t}, x_0, \lambda), \lambda) \geq a(\varphi(\tilde{t}, x_0, \lambda)) > R$ . This implies that there exists  $\tilde{t} < \tilde{t}$  such that  $V(\varphi(\tilde{t}, x_0, \lambda), \lambda) = R$  and  $V(\varphi(t, x_0, \lambda), \lambda) > R$  for  $t \in (\tilde{t}, \tilde{t})$ . Therefore for  $t \in (\tilde{t}, \tilde{t})$  one has  $b(\varphi(t, x_0, \lambda)) \geq V(\varphi(t, x_0, \lambda), \lambda) > R$ . Then  $\varphi(t, x_0, \lambda) \notin B_R$  for  $t \in (\tilde{t}, \tilde{t})$ . This is a contradiction, because  $-\dot{V}(\varphi(t, x_0, \lambda), \lambda) \geq c(\varphi(t, x_0, \lambda)) \geq 0$ , which implies that  $V(\varphi(t, x_0, \lambda), \lambda)$  is a decreasing function of  $t$  in this interval. Therefore  $\varphi(t, x_0, \lambda) \in A_R$  for  $t \in [0, t_+)$  which implies  $t_+ = \infty$ . The  $\omega$ -limit set of  $\varphi(t, x_0, \lambda)$  is contained in  $A_R$  and so  $\varphi(t, x_0, \lambda)$  tends to the largest invariant set of (2.2) contained in  $A_R$ , as  $t \rightarrow \infty$ .

(II) For  $x_0 \in \mathcal{B}_\rho - B_R$ , let  $[0, t_+)$  be the maximum interval of existence of the solution,  $\varphi(t, x_0, \lambda)$ , of (2.2). If there exists  $s \in (0, t_+)$  such that  $\varphi(s, x_0, \lambda) \in B_R$  then the problem is reduced to Part I.

Assuming that  $\varphi(t, x_0, \lambda) \notin B_R$ ,  $t \in [0, t_+)$ . If there exists  $\bar{t} \in (0, t_+)$  such that  $\varphi(\bar{t}, x_0, \lambda) \notin \mathcal{A}_\rho$  then  $\rho \leq a(\varphi(\bar{t}, x_0, \lambda)) \leq V(\varphi(\bar{t}, x_0, \lambda), \lambda)$  and  $V(\varphi(0, x_0, \lambda), \lambda) \leq b(\varphi(0, x_0, \lambda)) = b(x_0) < \rho$ , which leads to a contradiction, because outside of  $B_R$ ,  $\dot{V}(\varphi(t, x_0, \lambda), \lambda) \leq 0$ .

For  $t \in [0, t_+)$  one has  $a(\varphi(t, x_0, \lambda)) \leq V(\varphi(t, x_0, \lambda), \lambda) \leq V(\varphi(0, x_0, \lambda), \lambda) \leq b(\varphi(0, x_0, \lambda)) = b(x_0) < \rho$  and so  $\varphi(t, x_0, \lambda) \in \{x \in \mathbb{R}^n : a(x) \leq b(x_0)\}$ . Therefore  $t_+ = \infty$ . Letting  $\omega_\lambda$  be the  $\omega$ -limite set of  $\varphi(t, x_0, \lambda)$  then  $\omega_\lambda \subset \{x \in \mathbb{R}^n : a(x) \leq b(x_0)\}$ . As  $V(\varphi(t, x_0, \lambda), \lambda)$  is a decreasing and bounded function of  $t$  then there exists  $\lim_{t \rightarrow \infty} V(\varphi(t, x_0, \lambda), \lambda) := \ell \in \mathbb{R}$ . Then  $V(\cdot, \lambda) \equiv \ell$  and so  $\dot{V}(\cdot, \lambda) \equiv 0$  on  $\omega_\lambda$ . Since  $C \cap \omega_\lambda = \emptyset$  and  $\omega_\lambda \subset A_\rho$ , then  $0 = -\dot{V}(x, \lambda) \geq c(x) = 0$  for  $x \in \omega_\lambda$ . Thus  $\omega_\lambda \in E_\rho$ . This implies that  $\varphi(t, x_0, \lambda)$  tends to the largest invariant set of (2.2) contained in  $E_\rho$ , as  $t \rightarrow \infty$ . ■

*Remark 2.1.* If  $c(x) > 0$  for  $x \in \mathbb{R}^n - \bar{C}$ , or if for every  $x_0 \in E_\rho - \bar{C}$ ,  $\varphi(t, x_0, \lambda) \notin E_\rho$ , for every  $t > 0$  sufficiently small and the previous conditions of the theorem are satisfied, then we conclude that every solution, with initial condition in  $\mathcal{B}_\rho$ , tends to the largest invariant set contained in  $A_R$ , as  $t \rightarrow \infty$ . In this case, inside  $\mathcal{A}_\rho$ ,  $A_R$  will be a uniform estimate of the attractor and  $\mathcal{B}_\rho$  will be a uniform estimate of the basin of attraction.

*Remark 2.2.* The previous theorem can be reestablished, with appropriate modifications, to cover the case where  $\mathcal{A}_\rho$  is not bounded, but has a bounded connected component.

**THEOREM 2.4** (The global uniform invariance principle). *Suppose  $f: \mathbb{R}^n \times A \rightarrow \mathbb{R}^n$ ,  $V: \mathbb{R}^n \times A \rightarrow \mathbb{R}$  are  $C^1$  functions and that  $a, b, c: \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous functions. Suppose that*

$$a(x) \leq V(x, \lambda) \leq b(x), \quad -\dot{V}(x, \lambda) \geq c(x), \quad \forall (x, \lambda) \in \mathbb{R}^n \times A.$$

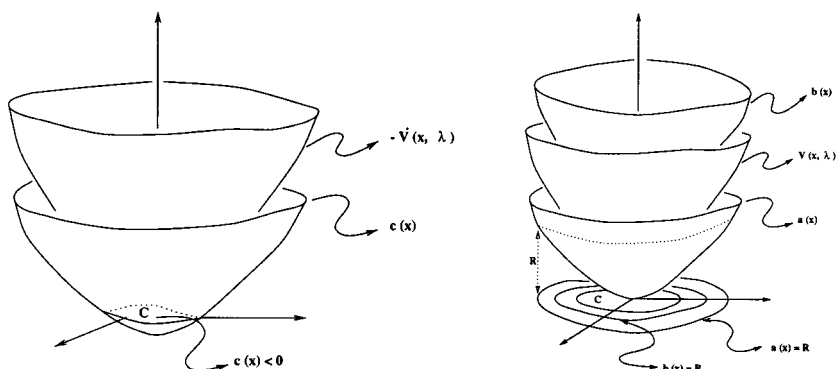
*Consider the sets:*

$$C := \{x \in \mathbb{R}^n : c(x) < 0\}, \quad E := \{x \in \mathbb{R}^n : c(x) = 0\}.$$

*Suppose that  $\sup_{x \in C} b(x) \leq R < \infty$  and consider the sets*

$$A_R := \{x \in \mathbb{R}^n : a(x) \leq R\}, \quad B_R := \{x \in \mathbb{R}^n : b(x) \leq R\}.$$

*Assume that  $A_R$  is nonempty and bounded (see Fig. 2).*

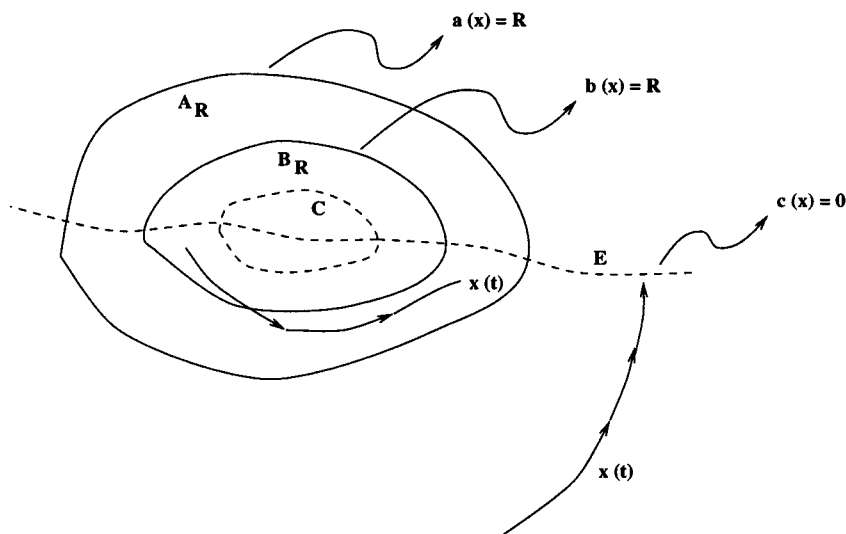


**FIG. 2.** The global uniform invariance principle. (a) Derivative of the Liapunov function, and (b) Liapunov function.

If  $\lambda$  is a fixed parameter in  $\Lambda$  and all the previous conditions are satisfied then the following holds:

(I) if  $x_0 \in B_R$  then  $\varphi(t, x_0, \lambda)$  is defined and belongs to  $A_R$ , for every  $t \geq 0$ , and tends to the largest invariant set of (2.2) contained in  $A_R$ , as  $t \rightarrow \infty$ .

(II) If  $x_0$  is such that the solution  $\varphi(t, x_0, \lambda)$  is bounded for  $t \geq 0$ , then  $\varphi(t, x_0, \lambda)$  tends to the largest invariant set of (2.2) contained in  $A_R \cup E$ , as  $t \rightarrow \infty$  (see Fig. 3).



**FIG. 3.**  $x(t)$  approaching the invariant set.

*Proof.* We consider the two above cases: Since  $C \subset B_R \subset A_R$  then  $\dot{V}(x, \lambda) \leq 0$ , for every  $(\lambda, x) \in A \times (\mathbb{R}^n - B_R)$ .

The proof of (I) is similar to the first part of the proof of the previous theorem.

To prove (II), we proceed as follows. If there exists  $t \geq 0$  such that  $\varphi(t, x_0, \lambda) \in B_R$ , the result follows from (I). Assuming that  $\varphi(t, x_0, \lambda)$  is bounded for  $t \geq 0$  and that  $\varphi(t, x_0, \lambda) \notin B_R$ , for every  $t \in [0, \infty)$ , then  $\dot{V}(\varphi(t, x_0, \lambda), \lambda) \leq 0$ , for every  $t \in [0, \infty)$ . Therefore,  $V(\varphi(t, x_0, \lambda), \lambda)$  is a decreasing function of  $t$ . Since  $V(\varphi(t, x_0, \lambda), \lambda)$  is bounded let  $\ell := \lim_{t \rightarrow \infty} V(\varphi(t, x_0, \lambda), \lambda)$ . Then  $V(\cdot, \lambda) \equiv \ell$  on the  $\omega$ -limit set,  $\omega_\lambda$ , of  $\varphi(t, x_0, \lambda)$  and so  $c(x) \leq -\dot{V}(x, \lambda) = 0$ , for  $x \in \omega_\lambda$ , which implies that  $c(x) = 0$ , for  $x \in \omega_\lambda$ , since  $C \cap \omega_\lambda = \emptyset$ . As  $\omega_\lambda$  is invariant with respect to (2.2), it is concluded that  $\varphi(t, x_0, \lambda)$  tends to the largest invariant set contained in  $E$ . ■

*Remark 2.3.* If  $a(x) \rightarrow \infty$ , as  $\|x\| \rightarrow \infty$ , then for every  $r > 0$  the set  $A_r := \{x \in \mathbb{R}^n : a(x) \leq r\}$  is bounded. If such condition is satisfied, then every solution is bounded for  $t \geq 0$  and the conclusion of the previous theorem holds true for every solution.

If  $c(x) > 0$  for every  $x \in \mathbb{R}^n - \bar{C}$ , or if for every  $x_0 \in E - \bar{C}$ ,  $\varphi(t, x_0, \lambda) \notin E$ , for every  $t > 0$  sufficiently small and the previous conditions of the theorem are satisfied, then it is possible to show that every solution tends to the largest invariant set contained in  $A_R$ , as  $t \rightarrow \infty$ . In such case one has that the set  $A_R$  is an estimate of the attractor and  $\mathbb{R}^n$  is the basin of attraction.

When using Theorems 2.3 or 2.4 in some applications, some technical difficulties may arise. The function  $c(x)$  may not be smooth, the set  $C$  may not be convex and so  $\sup_{x \in C} b(x)$  may not be attained on the boundary of the set  $C$ . Therefore the Lagrange multipliers technique cannot be used to compute  $\sup_{x \in C} b(x)$  even if  $b$  is a convex function. When some symmetries are present in the problem, the next lemma provides an alternative approach to avoid these difficulties.

**LEMMA 2.5.** *Let  $h, b, f_1, f_2, \dots, f_k: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous functions and assume that*

$$h(x) \geq \inf \{f_1(x), f_2(x), \dots, f_k(x)\}, \quad \forall x \in \mathbb{R}^n.$$

*Let*

$$F_i := \{x \in \mathbb{R}^n : f_i(x) < 0\}, \quad H := \{x \in \mathbb{R}^n : h(x) < 0\}.$$

Then the following hold:

- $H \subset \bigcup_{i=1}^k F_i$  and  $\sup_{x \in H} b(x) \leq \sup_{x \in \bigcup_{i=1}^k F_i} b(x)$ .

- Suppose  $F_i$  is bounded and that there exists a sequence of homeomorphisms  $S_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$  ( $i = 1, \dots, k$ ) such that  $F_j = S_{j-1}(F_{j-1})$ ,  $\forall j = 2, \dots, k$  and  $F_1 = S_k F_k$ . If  $b(S_i(x)) = b(x)$ ,  $\forall x \in \mathbb{R}^n$ ,  $\forall i = 1, \dots, k$  then  $\sup_{x \in \bigcup_{i=1}^k F_i} b(x) = \sup_{x \in F_j} b(x) \geq \sup_{x \in H} b(x)$ ,  $\forall j \in \{1, 2, \dots, k\}$ .

*Proof.* If  $x \in H$  then  $\inf \{f_1(x), f_2(x), \dots, f_k(x)\} \leq h(x) < 0$ . Therefore there exists  $j$  such that  $f_j(x) < 0$  and so  $x \in F_j \subset \bigcup_{i=1}^k F_i$ , which proves the first statement.

Next, one has to prove that  $\sup_{y \in F_{i-1}} b(y) = \sup_{x \in F_i} b(x)$ ,  $\forall i = 1, \dots, k$ . If  $y \in F_{i-1}$  there exists  $z \in F_i$  such that  $z = S_{i-1}(y)$  and so  $b(y) = b(S_{i-1}(y)) = b(z) \leq \sup_{x \in F_i} b(x)$ . Therefore  $\sup_{y \in F_{i-1}} b(y) \leq \sup_{x \in F_i} b(x)$ . As a consequence,

$$\sup_{x \in F_1} b(x) \leq \sup_{x \in F_2} b(x) \leq \dots \leq \sup_{x \in F_k} b(x) \leq \sup_{x \in F_1} b(x).$$

Therefore,  $\sup_{x \in F_i} b(x) = \sup_{x \in F_j} b(x)$  for any  $i, j \in \{1, \dots, k\}$  and so  $\sup_{x \in \bigcup_{i=1}^k F_i} b(x) = \sup_{x \in F_i} b(x)$ , for any  $i \in \{1, \dots, k\}$  and the proof is complete. ■

The following lemma has an obvious proof, but it is very useful to reduce the dimension in a problem of maximization. In fact, in the applications of this paper the reduction will be from dimension  $2n$  to dimension  $n$ .

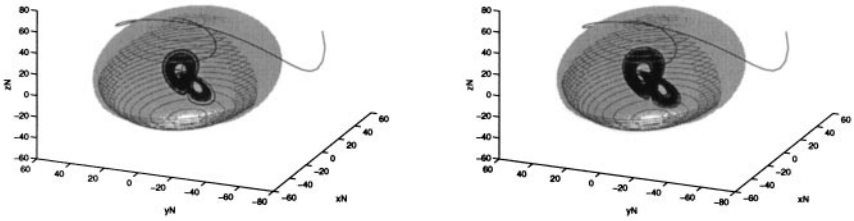
**LEMMA 2.6.** *Let  $A \subset \mathbb{R}^n$  be a compact set and  $b: A \rightarrow \mathbb{R}$  be a continuous function. Let  $D \subset \mathbb{R}^n$  a closed set such that  $A \cap D \neq \emptyset$  and for every  $x \in A$  there exists  $\bar{x} \in A \cap D$  such that  $b(\bar{x}) \geq b(x)$ . Then  $\sup_{x \in A} b(x) = \sup_{x \in A \cap D} b(x)$ .*

### 3. APPLICATIONS

**EXAMPLE 3.1** (Uniform estimate of the attractor of Lorenz System with parameter variation). Consider the Lorenz system:

$$\begin{cases} \dot{u} = -\sigma u + \sigma v \\ \dot{v} = -v - uz + ru \\ \dot{w} = -bw + uv. \end{cases}$$





**FIG. 4.** The uniform estimate of the Lorenz attractor. Initial condition:  $(x(0), y(0), z(0)) = (20, -70, 40)$ . (a)  $(x(t), y(t), z(t))$ ,  $(\sigma, r, b) = (9.5, 26.6, 2.53)$ ; (b)  $(x(t), y(t), z(t))$ ,  $(\sigma, r, b) = (10.5, 29.4, 2.8)$ .

With a change of variables in the previous system, as  $x := u$ ,  $y := v$ ,  $z := w - \frac{5}{4}r$ , the following system is obtained:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = -y - x(z + \frac{5}{4}r) + rx \\ \dot{z} = -b(z + \frac{5}{4}r) + xy. \end{cases} \quad (3.3)$$

The nominal values of the parameters are  $\sigma_N = 10$ ,  $r_N = 28$ , and  $b_N = \frac{8}{3}$ . An uncertainty of  $\pm 5\%$  is admitted to exist in the determination of these parameters (see Fig. 4).

Let  $\sigma_m := 9.5$ ,  $\sigma_M := 10.5$ ,  $r_m := 28 - \frac{28}{20}$ ,  $r_M := 28 + \frac{28}{20}$ ,  $b_m := \frac{8}{3} - \frac{8}{60}$ ,  $b_M := \frac{8}{3} + \frac{8}{60}$ .

Consider the set:

$$A := \{\lambda := (\sigma, r, b) \in \mathbb{R}^3 : \sigma_m \leq \sigma \leq \sigma_M, r_m \leq r \leq r_M, b_m \leq b \leq b_M\}.$$

If we consider the following Liapunov function for (3.3);

$$V(x, y, z) = rx^2 + 4\sigma y^2 + 4\sigma z^2,$$

we obtain

$$a(x, y, z) \leq V(x, y, z) \leq b(x, y, z);$$

where  $a(x, y, z) := r_m x^2 + 4\sigma_m y^2 + 4\sigma_m z^2$  and  $b(x, y, z) := r_M x^2 + 4\sigma_M y^2 + 4\sigma_M z^2$ .

If we estimate the derivative of  $V$  along the solutions of (3.3) we obtain:

$$\begin{aligned} -\dot{V}(r, \sigma, b, x, y, z) &= 2\sigma(rx^2 + 4y^2) + 8\sigma bz^2 + 10\sigma rbz \\ &\geq 2\sigma_m(r_m x^2 + 4y^2) + 8\sigma_m b_m z^2 - 10\sigma_M r_M b_M |z| \\ &= 2\sigma_m(r_m x^2 + 4y^2) + 8\sigma_m b_m \left[ z^2 - \frac{5\sigma_M r_M b_M}{4\sigma_m b_m} |z| \right] \end{aligned}$$

$$\begin{aligned}
&= 2\sigma_m(r_mx^2 + 4y^2) + 8\sigma_mb_m \left[ z^2 - 2 \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} |z| \right. \\
&\quad \left. + \left( \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} \right)^2 - \left( \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} \right)^2 \right] \\
&= 2\sigma_m(r_mx^2 + 4y^2) + 8\sigma_mb_m \left[ z^2 - 2 \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} |z| \right. \\
&\quad \left. + \left( \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} \right)^2 \right] - 8\sigma_mb_m \left( \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} \right)^2 \\
&= 2\sigma_m(r_mx^2 + 4y^2) + 8\sigma_mb_m \left( |z| - \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} \right)^2 - \frac{(5\sigma_M r_M b_M)^2}{8\sigma_m b_m} \\
&:= c(x, y, z) := \alpha x^2 + \beta y^2 + \gamma(|z| - \rho)^2 - \mu
\end{aligned}$$

The above expression naturally defines the numbers  $\alpha, \beta, \gamma$ .

Now we will use Lemma 2.5 with  $h = c$ ,  $f_1(x, y, z) := \alpha x^2 + \beta y^2 + \gamma(z - \rho)^2 - \mu$  and  $f_2(x, y, z) := \alpha x^2 + \beta y^2 + \gamma(z + \rho)^2 - \mu$ .

If we let  $C := \{(x, y, z) \in \mathbb{R}^3 : c(x, y, z) < 0\}$ ,  $F_1 := \{(x, y, z) \in \mathbb{R}^3 : f_1(x, y, z) < 0\}$  from Lemma 2.5 it follows that  $\sup_C b \leq \sup_{F_1} b$ .

Using the Lagrange function

$$\begin{aligned}
\mathcal{L}(x, y, z, \mu) &= r_M x^2 + 4\sigma_M y^2 + 4\sigma_M z^2 \\
&+ \mu \left[ 2\sigma_m r_m x^2 + 8\sigma_m y^2 + 8\sigma_m b_m \left( z - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right)^2 - \frac{(5\sigma_M b_M r_M)^2}{8\sigma_m b_m} \right]
\end{aligned}$$

the following extreme conditions are obtained:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x(r_M + 2\mu\sigma_m r_m) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 8y(\sigma_M + 2\mu\sigma_m) = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 8\sigma_M z + 16\mu\sigma_m b_m \left( z - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 2\sigma_m(r_mx^2 + 4y^2) + 8\sigma_mb_m \left( z - \frac{5\sigma_M r_M b_M}{8\sigma_m b_m} \right)^2 - \frac{(5\sigma_M r_M b_M)^2}{8\sigma_m b_m} = 0.$$

The maximum is attained at  $x=0$ ,  $y^2=(25\sigma_M^2b_M^2r_M^2(b_m-2))/(64\sigma_m(1-b_m)^2)$ ,  $z=(5\sigma_Mb_Mr_M)/(8\sigma_m(b_m-1))$

$$\sup_{F_1} b = \frac{25\sigma_M^3b_M^2r_M^2}{16\sigma_m^2(b_m-1)} = \frac{3^6 \times 7^9}{2^3 \times 5 \times 19^2 \times 23} < 88575, \quad 75 < R := 88576.$$

Therefore the Lorenz attractor is contained in the ellipsoid:

$$\{(x, y, z) \in \mathbb{R}^3 : a(x, y, z) = r_mx^2 + 4\sigma_my^2 + 4\sigma_mz^2 < R = 88576\}.$$

The above estimation as well an idea of the Lorenz attractor is shown in Fig. 4, where the external ellipsoid corresponds to the set  $A_R$  and the internal ellipsoid corresponds to the set  $B_R$ .

In what follows, the invariance principle will be used to study the synchronization of several systems. First of all let us define formally the concept of synchronization used in this paper.

This paper is concerned with the synchronization of two coupled systems of the following form,

$$\begin{cases} \dot{x} = f(x, y, \lambda_1) \\ \dot{y} = g(x, y, \lambda_2), \end{cases} \quad (3.4)$$

where  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ .

Let  $A$  be a nonempty open subset of  $\mathbb{R}^n \times \mathbb{R}^n$ ,  $M \subset A$  a  $C^1$ -manifold and  $d(\cdot, \cdot)$  a distance in  $\mathbb{R}^n \times \mathbb{R}^n$ .

**DEFINITION 3.1.** System (3.4) synchronizes with respect to  $A$  and  $M$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that, if  $\|\lambda_1 - \lambda_2\| < \delta$ , then

$$\limsup_{t \rightarrow \infty} d((x(t, x_o, y_o), y(t, x_o, y_o)), M) \leq \varepsilon$$

for all initial conditions  $(x_o, y_o) \in A$ .

If  $A = \mathbb{R}^n \times \mathbb{R}^n$ , we say that the system (3.4) synchronizes globally.

In many applications, the synchronization is global and the manifold  $M$  is the diagonal  $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ . In that case the above definition may be stated as follows:

**DEFINITION 3.2.** System (3.4) synchronizes globally, with respect to the diagonal, if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that, if  $\|\lambda_1 - \lambda_2\| < \delta$ , then

$$\limsup_{t \rightarrow \infty} \|y(t, x_o, y_o, \lambda_1, \lambda_2) - x(t, x_o, y_o, \lambda_1, \lambda_2)\| \leq \varepsilon$$

for all initial conditions  $(x_o, y_o) \in \mathbb{R}^n \times \mathbb{R}^n$ .

See Hale [10], for related definitions.

EXAMPLE 3.2 (Reduced coupled power systems). In this first example, the usual invariance principle will be used to study the synchronization of a reduced model [5, 6] associated to a two-machine-infinite-bus power system. This system is represented by two differential equations coupled through a nonlinear function:

$$\begin{cases} \dot{x} = p_1 - c_1 \sin x - k \sin(x - y) \\ \dot{y} = p_2 - c_2 \sin y - k \sin(y - x). \end{cases} \quad (3.5)$$

Consider the following Liapunov function for (3.5).

$$V(x, y) = -p_1 x - p_2 y + c_1(1 - \cos x) + c_2(1 - \cos y) + k(1 - \cos(x - y)) \quad (3.6)$$

Computing the derivative of the previous function along the solution of the system, one obtains

$$-\dot{V}(x, y) = [p_1 - c_1 \sin x - k \sin(x - y)]^2 + [p_2 - c_2 \sin y - k \sin(y - x)]^2.$$

Therefore  $-\dot{V}(x, y) = 0$  if and only if

$$p_1 - c_1 \sin x = k \sin(x - y)$$

$$p_2 - c_2 \sin y = k \sin(y - x).$$

It is interesting to study the previous system for  $p_1$  close to  $p_2$  and  $c_1 > 0$  close to  $c_2 > 0$ . Then one has:

$$\begin{cases} p_1 - c_1 \sin x = k \sin(x - y) \\ p_1 - c_1 \sin y = k \sin(y - x) + [(p_1 - p_2) + (c_2 - c_1) \sin y] \end{cases}$$

$$\begin{cases} \frac{p_1}{c_1} - \sin x = \frac{k}{c_1} \sin(x - y) \\ \frac{p_1}{c_1} - \sin y = \frac{k}{c_1} \sin(y - x) + \frac{1}{c_1} [(p_1 - p_2) + (c_2 - c_1) \sin y] \end{cases}$$

Now letting  $a := p_1/c_1$ ,  $K := k/c_1$  and  $h := 1/c_1 [(p_1 - p_2) + (c_2 - c_1) \sin y]$  one obtains:

$$\begin{cases} a - \sin x = K \sin(x - y) \\ a - \sin y = K \sin(y - x) + h \end{cases}$$

Adding and subtracting the previous expressions, one gets

$$\begin{cases} \sin x + \sin y = 2a - h \\ \sin x - \sin y = -2K \sin(x - y) + h \end{cases}$$

which is equivalent to:

$$\begin{cases} \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = a - \frac{h}{2} \\ \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right) + 2K \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x-y}{2}\right) = \frac{h}{2}. \end{cases}$$

Now assuming that  $p_1$  is sufficiently close to  $p_2$  and  $c_1$  sufficiently close to  $c_2$ , in such a way that  $|\frac{h}{2}| < \varepsilon < a$ , the previous system becomes equivalent to

$$\begin{cases} \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = a - \frac{h}{2} \\ \sin\left(\frac{x-y}{2}\right) \left[ \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x+y}{2}\right) + 2K \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \right] = \frac{h}{2} \sin\left(\frac{x+y}{2}\right) \end{cases} \quad (3.7)$$

and so,

$$\begin{cases} \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = a - \frac{h}{2} \\ \sin\left(\frac{x-y}{2}\right) \left[ \sin(x+y) + 4K\left(a - \frac{h}{2}\right) \right] = 2\frac{h}{2} \sin\left(\frac{x+y}{2}\right). \end{cases} \quad (3.8)$$

Since  $|\sin(x+y) + 4K(a - \frac{h}{2})| \geq 4K(a - \varepsilon) - 1$ , assuming that  $K > \frac{1}{a-\varepsilon}$ , one obtains:

$$\begin{aligned} (4K(a - \varepsilon) - 1) \left| \sin\left(\frac{x-y}{2}\right) \right| &\leq \left| \sin\left(\frac{x-y}{2}\right) \left[ \sin(x+y) + 4K\left(a - \frac{h}{2}\right) \right] \right| \\ &= \left| 2\frac{h}{2} \sin\left(\frac{x+y}{2}\right) \right| \leq 2\varepsilon. \end{aligned}$$

For  $|\frac{x-y}{2}| < \frac{3\pi}{4}$  or equivalently,  $|x-y| < \frac{3\pi}{2}$ , one has

$$\begin{aligned} & \frac{2\sqrt{2}}{3\pi} (4K(a-\varepsilon)-1) \left| \frac{x-y}{2} \right| \\ & \leq \left| \sin\left(\frac{x-y}{2}\right) \left[ \sin(x+y) + 4K\left(a-\frac{h}{2}\right) \right] \right| \leq 2\varepsilon. \end{aligned}$$

Therefore for  $|x-y| < \frac{3\pi}{2}$ , one obtains the following a priori estimate for the equilibria:

$$|x-y| \leq \frac{\varepsilon}{K} \frac{6\pi}{\sqrt{2} [4(a-\varepsilon) - (1/K)]}. \quad (3.9)$$

In terms of the original parameters the conditions,  $a > \varepsilon > \frac{h}{2}$ ,  $K > \frac{1}{4(a-\varepsilon)}$  will be satisfied if  $p_1/c_1 > \varepsilon > 1/c_1 [|p_1 - p_2| + |c_1 - c_2|]$  and  $k > c_1^2/4(p_1 - \varepsilon c_1)$ , respectively.

In order to study the behavior of the equilibria, when  $K$  is large or  $\varepsilon$  is small, let us take  $\lambda_i := (c_i, p_i)$ ,  $i = 1, 2$  and  $\mu := 1/K$ . From Eq. (3.7) one obtains:

$$\begin{cases} \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) = a - \frac{h}{2} \\ \sin\left(\frac{x-y}{2}\right) \left[ \left(a - \frac{h}{2}\right) + \mu \frac{\sin(x+y)}{4} \right] = \mu \frac{h}{4} \sin\left(\frac{x+y}{2}\right). \end{cases} \quad (3.10)$$

Letting  $w := \frac{x-y}{2}$  the second equation of the previous system becomes equivalent to  $f(w, y, \mu, \lambda_1, \lambda_2) := \sin(w) [(a - \frac{h}{2}) + \mu (\sin 2(w+y))/4] - \mu \frac{h}{4} \sin(y+w) = 0$ .

Using the implicit function theorem, the previous equation can be solved to obtain  $w = w(\mu, y, \lambda_1, \lambda_2) = O(\mu |\lambda_1 - \lambda_2|)$ , or  $x = y + O(\mu |\lambda_1 - \lambda_2|)$ . Substituting this last expression in the first equation of (3.10), one obtains

$$\sin(y + O(\mu |\lambda_1 - \lambda_2|)) \cos(O(\mu |\lambda_1 - \lambda_2|)) = a - \frac{h}{2} \quad (3.11)$$

which is equivalent to:

$$\sin y - a + \frac{h}{2} = O(\mu |\lambda_1 - \lambda_2|) \quad (3.12)$$

Letting  $\mu = 0$  in the previous equation, one obtains:  $\sin y - a + \frac{h}{2} = 0$  which is equivalent to:

$$\sin y = \frac{p_1 + p_2}{c_1 + c_2}. \quad (3.13)$$

Considering  $0 < (p_1 + p_2)/(c_1 + c_2) < 1$  fixed, it is possible to find  $y_i \in [0, 2\pi A, i = 1, 2$  such that  $\sin y_i = (p_1 + p_2)/(c_1 + c_2)$  AAAAAAB, and from the function theorem it follows that Eq. (3.12) can be solved to obtain  $y = y_i + O(\mu)$ , for  $\mu$  sufficiently small or, in terms of the original parameters,  $y = y_i + O(\frac{1}{k})$ , for  $k$  sufficiently large.

Now assuming that  $k > c_1^2/(4(p_1 - \varepsilon c_1))$ , for  $|\lambda_1 - \lambda_2|$ , sufficiently small, Eq. (3.12) is equivalent to:

$$\sin y = \frac{p_1}{c_1} + O(|\lambda_1 - \lambda_2|). \quad (3.14)$$

For  $0 < p_1/c_1 < 1$  fixed, it is possible to find  $y_i \in [0, 2\pi], i = 1, 2$ , such that  $\sin y_i = p_1/c_1$  and from the implicit function theorem it follows that Eq. (3.14) can be solved to obtain  $y = y_i + O(|\lambda_1 - \lambda_2|)$ , for  $|\lambda_1 - \lambda_2|$  sufficiently small.

As a consequence it is possible to conclude that the original system synchronizes either if  $|\lambda_1 - \lambda_2| \rightarrow 0$  or if  $k \rightarrow \infty$ .

From the invariance principle it follows that every solution, which is bounded for  $t \geq 0$ , tends to an equilibrium point, as  $t \rightarrow \infty$ .

**EXAMPLE 3.3** (Coupled power systems). The following differential equations are derived from power system studies. These equations represent the dynamical behavior of two synchronous machines versus an infinite bus.

$$\begin{cases} \dot{x} = y \\ \dot{y} = p - a \sin x - K \sin(x - u) - by - k(y - v) \\ \dot{u} = v \\ \dot{v} = P - A \sin u - K \sin(u - x) - Bv - k(v - y), \end{cases} \quad (3.15)$$

where  $x$  and  $u$  are the rotor angles of the machines and  $y$  and  $v$  are the angular velocities.

From the applied point of view, it is interesting to identify coherent machines, that is, synchronized machines. Therefore, sufficient conditions to obtain the synchronization will be given. Consider the following Liapunov function:

$$\begin{aligned} V(x, y, u, v) := & \frac{y^2}{2} + \frac{v^2}{2} - px - Pu + a(1 - \cos x) \\ & + A(1 - \cos u) + K(1 - \cos(x - u)) \end{aligned}$$

The derivative of  $V$  along the solutions is given by:

$$\dot{V}(x, y, u, v) = -by^2 - Bv^2 - k(y - v)^2 \leq 0$$

If  $b$  and  $B$  are positive, the set in which the derivative of  $V$  is zero is the plane  $y = v = 0$  and then every solution that is bounded for  $t \geq 0$  must approach an equilibrium point as  $t \rightarrow \infty$ . These equilibria were studied in the previous example and they are located on the sets  $\Gamma_n := \{(x, 0, u, 0) : x - u = 2n\pi\}$ , where  $n$  is a nonnegative integer, when the subsystems are identical. This follows from the invariance principle.

When the two subsystems are not identical, but  $p - P$ ,  $a - A$ , and  $b - B$  are small, the solutions that are bounded for  $t \geq 0$  approach equilibria located in a set that is close to  $\Gamma_n$ , as  $t \rightarrow \infty$ .

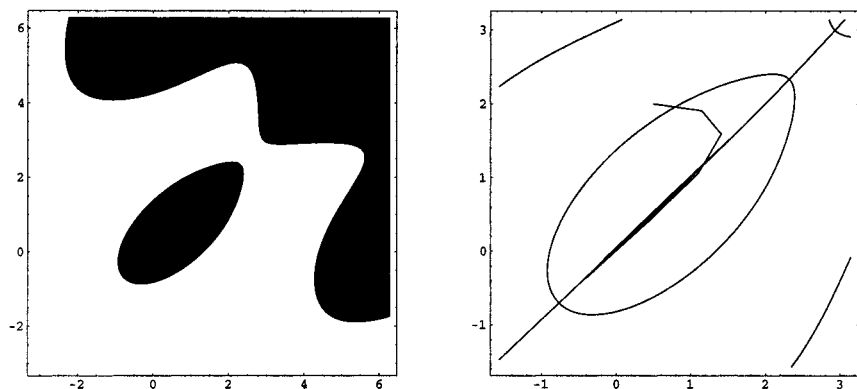
Consider now the case  $b = B = 0$ . In this case if  $k > 0$  the derivative is zero if and only if  $y = v$ . Therefore, on the largest invariant set contained in the set  $E$ ,  $\dot{y} - \dot{v} = 0$ , that is:

$$(p - P) - a \sin x + A \sin u - 2K \sin(x - u) = 0. \quad (3.16)$$

The previous Liapunov function can also be used to obtain bounded positive invariant sets for the coupled system. For example, consider Eq. (3.16) for  $p = 0.3$ ,  $P = 0.4$ ,  $a = 0.7$ ,  $A = 0.8$ ,  $K = 1$ , and  $k = 1$ . One can consider the set

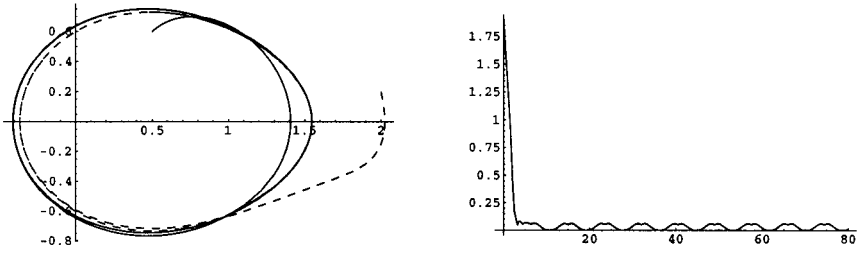
$$\Omega_{0.9} := \{(x, y, u, v) : V(x, y, u, v) < 0.9\}$$

that set has a bounded component. Figure 5a shows the set  $\{(x, u) \in [-\pi, 2\pi] \times [-\pi, 2\pi] : V(x, 0, u, 0) < 0.9\}$ . Figure 5b shows the solution



**FIG. 5.** Power system: Invariant set. (a) Projection of a positively invariant set. (b) Solutions approaching a set close to the diagonal.





**FIG. 6.** Synchronization of power systems. (a)  $(x(t), y(t))$ ,  $(u(t), v(t))$ ,  $t \in [0, 20]$ . (b)  $|x(t) - y(t)| + |u(t) - v(t)|$ ,  $t \in [0, 80]$ .

$(x(t), u(t))$  of (3.16), with initial condition  $x(0) = 0.5$ ,  $y(0) = 0.6$ ,  $u(0) = 2$ , and  $v(0) = 0.2$  approaching the projection of a manifold whose points  $(x, y, u, v)$  satisfy

$$(p - P) - a \sin x + A \sin u - 2K \sin(x - u) = 0, \quad y = v$$

and is close to the diagonal. Figure 6 indicates synchronization.

**EXAMPLE 3.4** (Uniform estimate of the attractor and synchronization of coupled Lorenz System with parameter variation). Consider two Lorenz systems coupled through a linear term:

$$\begin{cases} \dot{u}_1 = -\sigma_1 u_1 + \sigma_1 v_1 - k(u_1 - u_2) \\ \dot{v}_1 = -v_1 - u_1 w_1 + r_1 u_1 \\ \dot{w}_1 = -b_1 w_1 + u_1 v_1 \\ \dot{u}_2 = -\sigma_2 u_2 + \sigma_2 v_2 - k(u_2 - u_1) \\ \dot{v}_2 = -v_2 - u_2 w_2 + r_2 u_2 \\ \dot{w}_2 = -b_2 w_2 + u_2 v_2 \end{cases} \quad (3.17)$$

The nominal values of the parameters are  $\sigma_N = 10$ ,  $r_N = 28$ , and  $b_N = \frac{8}{3}$ . Allowing an uncertainty of  $\pm 5\%$  on the determination of these parameters, we define  $\sigma_m := 9.5$ ,  $\sigma_M := 10.5$ ,  $r_m := 28 - \frac{28}{20}$ ,  $r_M := 28 + \frac{28}{20}$ , and  $b_m := \frac{8}{3} - \frac{8}{60}$ ,  $b_M := \frac{8}{3} + \frac{8}{60}$ . Consider the following set,

$$A := \{\lambda \in \mathbb{R}^6 : \sigma_m \leq \sigma_1, \sigma_2 \leq \sigma_M, r_m \leq r_1, r_2 \leq r_M, b_m \leq b_1, b_2 \leq b_M\},$$

where  $\lambda := (\sigma_1, r_1, b_1, \sigma_2, r_2, b_2)$ .

Our purpose is the study of synchronization of this system. However let us first consider a simpler case, where two identical Lorenz systems,

without the nonlinear terms, are coupled and with parameters  $\sigma := \sigma_N = 10$ ,  $r := r_N = 28$  and  $b := b_N = \frac{8}{3}$ .

$$\begin{cases} \dot{u}_1 = -\sigma u_1 + \sigma v_1 - k(u_1 - u_2) \\ \dot{v}_1 = -v_1 + r u_1 \\ \dot{w}_1 = -b w_1 \\ \dot{u}_2 = -\sigma u_2 + \sigma v_2 - k(u_2 - u_1) \\ \dot{v}_2 = -v_2 + r u_2 \\ \dot{w}_2 = -b w_2 \end{cases} \quad (3.18)$$

Considering the difference between the components of (3.18), one obtains:

$$\frac{d}{dt} \begin{pmatrix} u_2 - u_1 \\ v_2 - v_1 \\ w_2 - w_1 \end{pmatrix} = \begin{bmatrix} -(\sigma + 2k) & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{pmatrix} u_2 - u_1 \\ v_2 - v_1 \\ w_2 - w_1 \end{pmatrix}. \quad (3.19)$$

The eigenvalues of the previous matrix are given by:

$$-\frac{8}{3}, \text{ and } \frac{-(29 + 2k) \pm \sqrt{(29 + 2k)^2 - 4(2k - 270)}}{2}.$$

Therefore all eigenvalues will have negative real parts if and only if  $k > 135$  and so system (3.18) will synchronize if and only if  $k > 135$ .

Let us consider now the more general case concerning system (3.17).

With a change of variables in the previous system, as  $x_i := u_i$ ,  $y_i := v_i$ , and  $z_i := w_i - \frac{5}{4} r_i$ , the following system is obtained:

$$\begin{cases} \dot{x}_1 = -\sigma_1 x_1 + \sigma_1 y_1 - k(x_1 - x_2) \\ \dot{y}_1 = -y_1 - x_1(z_1 + \frac{5}{4} r_1) + r_1 x_1 \\ \dot{z}_1 = -b_1(z_1 + \frac{5}{4} r_1) + x_1 y_1 \\ \dot{x}_2 = -\sigma_2 x_2 + \sigma_2 y_2 - k(x_2 - x_1) \\ \dot{y}_2 = -y_2 - x_2(z_2 + \frac{5}{4} r_2) + r_2 x_2 \\ \dot{z}_2 = -b_2(z_2 + \frac{5}{4} r_2) + x_2 y_2 \end{cases}$$

Let

$$\begin{aligned} V(x_1, y_1, z_1, x_2, y_2, z_2, \lambda) \\ = x_1^2 + x_2^2 + 4 \frac{\sigma_1}{r_1} y_1^2 + 4 \frac{\sigma_2}{r_2} y_2^2 + 4 \frac{\sigma_1}{r_1} z_1^2 + 4 \frac{\sigma_2}{r_2} z_2^2 \end{aligned} \quad (3.20)$$

be a Liapunov function for the previous system.

Our next purpose is to show that the conditions of Remark 2.3 and Theorem 2.4 are satisfied.

For the functions defined below, one has:

$$a(x_1, y_1, z_1, x_2, y_2, z_2) := x_1^2 + x_2^2 + 4 \frac{\sigma_m}{r_M} (y_1^2 + y_2^2) + 4 \frac{\sigma_m}{r_M} (z_1^2 + z_2^2)$$

$$b(x_1, y_1, z_1, x_2, y_2, z_2) := x_1^2 + x_2^2 + 4 \frac{\sigma_M}{r_m} (y_1^2 + y_2^2) + 4 \frac{\sigma_M}{r_m} (z_1^2 + z_2^2)$$

$$a(x_1, y_1, z_1, x_2, y_2, z_2) \leq V(x_1, y_1, z_1, x_2, y_2, z_2, \lambda)$$

$$\leq b(x_1, y_1, z_1, x_2, y_2, z_2), \quad \forall \lambda \in A$$

The derivative of  $V$  is given by

$$\begin{aligned} & -\dot{V}(x_1, y_1, z_1, x_2, y_2, z_2, \lambda, k) \\ &= 2\sigma_1 \left( x_1^2 + \frac{4}{r_1} y_1^2 + 4 \frac{b_1}{r_1} z_1^2 + 5b_1 z_1 \right) \\ & \quad + 2\sigma_2 \left( x_2^2 + \frac{4}{r_2} y_2^2 + 4 \frac{b_2}{r_2} z_2^2 + 5b_2 z_2 \right) + 2k(x_1 - x_2)^2 \\ & \geq 2\sigma_m(x_1^2 + x_2^2) + \frac{8\sigma_m}{r_M} (y_1^2 + y_2^2) + 8 \frac{\sigma_m b_m}{r_M} \left[ \left( |z_1|^2 - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right)^2 \right. \\ & \quad \left. + \left( |z_2|^2 - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right)^2 \right] - \frac{(5\sigma_M b_M)^2}{4\sigma_m b_m} r_M \\ & := \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[ (|z_1| - \rho)^2 + (|z_2| - \rho)^2 ] - \mu \\ & := c(x_1, y_1, z_1, x_2, y_2, z_2), \end{aligned}$$

for every  $\lambda \in A$  and  $k > 0$ . The previous identities also define the parameters  $\alpha, \beta, \gamma, \rho, \mu$ .

Note that the functions  $a, b$  previously obtained are regular functions. However, the function  $c$  is not regular and the set where  $c < 0$  is not convex, which brings some technical difficulties in the application of the Lagrange multipliers technique. In order to overcome this difficulty, Lemma 2.5 will be used.

Consider  $f_1, f_2, f_3, f_4: \mathbb{R}^6 \rightarrow \mathbb{R}$  defined as:

$$\begin{aligned}
 f_1(x_1, y_1, z_1, x_2, y_2, z_2) \\
 &:= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[(z_1 - \rho)^2 + (z_2 - \rho)^2] - \mu \\
 f_2(x_1, y_1, z_1, x_2, y_2, z_2) \\
 &:= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[(z_1 + \rho)^2 + (z_2 - \rho)^2] - \mu \\
 f_3(x_1, y_1, z_1, x_2, y_2, z_2) \\
 &:= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[(z_1 + \rho)^2 + (z_2 + \rho)^2] - \mu \\
 f_4(x_1, y_1, z_1, x_2, y_2, z_2) \\
 &:= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[(z_1 - \rho)^2 + (z_2 + \rho)^2] - \mu.
 \end{aligned}$$

For  $F_i := \{x \in \mathbb{R}^6 : f_i(x) < 0\}$ , the homeomorphisms  $S_i$  can be given explicitly in a natural way, for  $i = 1, \dots, 4$ . In fact in this case they will be linear involutions.

It is easy to see that  $c \geq \inf_{i=1, \dots, 4} \{f_i\}$ . Letting  $C := \{x \in \mathbb{R}^6 : c(x) < 0\}$ , from Lemma 2.5 it follows that  $\sup_{x \in C} b(x) \leq \sup_{x \in F_1} b(x)$ .

Now we reduce the problem from  $\mathbb{R}^6$  to  $\mathbb{R}^3$  using Lemma 2.6. In this case the set  $F_1$  plays the role of the set  $A$  of Lemma 2.6. The set  $D := \{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : x_1 = x_2, y_1 = y_2, z_1 = z_2\}$  is the diagonal.

For  $(x_1, y_1, z_1, x_2, y_2, z_2) \in F_1$ , define

$$\bar{x} = \sqrt{\frac{x_1^2 + x_2^2}{2}}, \quad \bar{y} = \sqrt{\frac{y_1^2 + y_2^2}{2}}, \quad \text{and} \quad \bar{z} = \sqrt{\frac{z_1^2 + z_2^2}{2}}$$

and take  $(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}) \in D$ .

It is clear that  $b(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}) \geq b(x_1, y_1, z_1, x_2, y_2, z_2)$ . Therefore it is necessary to prove that  $(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}) \in A \cap D$ , that is, it is necessary to prove that  $(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}) \in F_1$ .

$$\begin{aligned}
 f_1(\bar{x}, \bar{y}, \bar{z}, \bar{x}, \bar{y}, \bar{z}) &:= \alpha(2\bar{x}^2) + \beta(2\bar{y}^2) + \gamma[2(\bar{z} - \rho)^2] - \mu \\
 &= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + 2\gamma \left( \sqrt{\frac{z_1^2 + z_2^2}{2}} - \rho \right)^2 - \mu \\
 &= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma(z_1^2 + z_2^2 - \sqrt{z_1^2 + z_2^2} \rho + 2\rho^2) - \mu \\
 &\leq \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma(z_1^2 + z_2^2 - 2z_1\rho - 2z_2\rho + 2\rho^2) - \mu \\
 &= \alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[(z_1 - \rho)^2 + (z_2 - \rho)^2] - \mu < 0.
 \end{aligned}$$

As a consequence,  $(\bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{z}, \bar{z}) \in F_1$  as required. Therefore, the problem of computing the supremum can be reduced to half of the dimension of the original problem, that is:

$$\sup_{x \in F_1 \cap D} b(x) = \sup_{x \in F_1} b(x).$$

As  $F_1 \cap D$  is a convex set and  $b$  is a convex function, the sup of  $b$  in  $F_1 \cap D$  is attained on the boundary of  $F_1 \cap D$ . Therefore the techniques of Lagrange multipliers could be used to compute the sup of  $b$  in the set  $F_1 \cap D$ , taking the boundary of the set  $F_1 \cap D$  as a constraint equation.

Using the Lagrange function

$$\begin{aligned} \mathcal{L}(x, y, z, \mu) = & 2x^2 + \frac{8\sigma_M}{r_m}(y^2) + \frac{8\sigma_M}{r_m}(z^2) + \mu \left[ 4\sigma_m(x^2) + \frac{16\sigma_m}{r_M}(y^2) \right. \\ & \left. + 16 \frac{\sigma_m b_m}{r_M} \left[ \left( z - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right)^2 + \right] - \frac{(5\sigma_M b_M)^2}{4\sigma_m b_m} r_M \right] \end{aligned}$$

the following extreme conditions are obtained:

$$\frac{\partial \mathcal{L}}{\partial x} = 4x(1 + 2\mu\sigma_m) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 16y \left( \frac{\sigma_M}{r_m} + 2\mu \frac{\sigma_m}{r_M} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 16 \frac{\sigma_M}{r_m} z + \frac{32\mu\sigma_m b_m}{r_M} \left( z - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right) = 0$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mu} = & 4\sigma_m(x^2) + \frac{16\sigma_m}{r_M}(y^2) + 16 \frac{\sigma_m b_m}{r_M} \left[ \left( z - \frac{5\sigma_M b_M r_M}{8\sigma_m b_m} \right)^2 \right] \\ & - \frac{(5\sigma_M b_M)^2}{4\sigma_m b_m} r_M = 0. \end{aligned}$$

The solution of the previous system is  $\mu = (2\sigma_M r_M)/(\sigma_m b_m(2r_m - 1))$ ,  $x = y = 0$ , and  $z = (10\sigma_m r_M b_M r_m)/(4\sigma_m b_m(2r_m + 3))$ . Substituting these values in the expression of  $b$ , the number  $R$  is obtained:

$$R = \sup_{(x, y, z) \in F} b(x) = \frac{50\sigma_M^3 r_M^2 b_M^2 r_m}{\sigma_m^2 b_m^2 (2r_m + 3)^2} = \frac{3^7 \times 5 \times 7^{10}}{19^3 \times 281} < 5703.29 < 5703.3.$$

The set  $B_R$  is the ellipsoid:

$$\{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : b(x_1, y_1, z_1, x_2, y_2, z_2) \leq R\}$$

The set  $A_R$  is also an ellipsoid which contains the set  $B_R$ :

$$\{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : a(x_1, y_1, z_1, x_2, y_2, z_2) \leq R\}.$$

The set in which  $c(x_1, y_1, z_1, x_2, y_2, z_2) = 0$  is contained in  $A_R$  and so every solution converges to the largest invariant set contained in  $A_R$ . The set  $A_R$  is an estimate of the attractor which is independent of the parameters  $\lambda \in \Lambda$  and  $k > 0$ . Therefore every solution of (3.17) enters in  $A_R$  in finite time and stays there in the future.

In order to study the synchronization, either Theorem 2.3 of [20] or Theorem 3.1 of [1] can be used. With this aim, it is convenient to rewrite (3.17) system in the following form,

$$\dot{u} = -A_k(u - v) + f(u, \lambda_1)$$

$$\dot{v} = -A_k(v - u) + f(v, \lambda_2),$$

where  $A_k$  is a constant matrix which represents the coupling between the Lorenz systems,  $\lambda_1 = (\sigma_1, r_1, b_1)$  is the parameter vector of the first system,  $\lambda_2 = (\sigma_2, r_2, b_2)$  is the parameter vector of the second system,  $u = (x_1, y_1, z_1)^T$  is the state vector of the first system, and  $v = (x_2, y_2, z_2)^T$  is the state vector of the second system. In this case:

$$A_k = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad f(\cdot, \lambda) = \begin{bmatrix} -\sigma x + \sigma y \\ -y - x(z + \frac{5}{4}r) + rx \\ -b(z + \frac{5}{4}r) + xy \end{bmatrix}.$$

It is easy to see that  $f(v, \lambda_2) - f(v, \lambda_1) = O(|\lambda_2 - \lambda_1|)$ , recalling that  $v$  stays in a bounded set.

Note that  $f(v, \lambda) - f(u, \lambda)$  can be rewritten in the following form,

$$f(v, \lambda) - f(u, \lambda) = F(u, v, \lambda_2)[v - u],$$

where

$$F(u, v, \lambda) = \begin{bmatrix} -\sigma & +\sigma & 0 \\ -\frac{1}{4}r - z_1 & -1 & -x_2 \\ y_1 & x_2 & -b \end{bmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}.$$

Following Theorem 2.3 of [20] or Theorem 3.1 of [1], an exponential decay for the evolutions operator of

$$\dot{\xi} = \begin{pmatrix} -\sigma - 2k & +\sigma & 0 \\ -\frac{1}{4}r - z_1 & -1 & -x_2 \\ y_1 & x_2 & -b \end{pmatrix} \xi \quad (3.21)$$

must be obtained. With that in mind, consider the following Liapunov function:

$$W(\xi_1, \xi_2, \xi_3) := \frac{1}{2} [(\xi_1)^2 + \beta(\xi_2)^2 + \beta(\xi_3)^2]$$

Computing the derivative of  $W$ , with respect to (3.24), one obtains, for  $\beta := 0.1$ :

$$-\dot{W} = (\sigma + 2k)(\xi_1)^2 - (\sigma - \frac{1}{4}\beta r - \beta z_1) \xi_1 \xi_2 + \beta(z_2)^2 + \beta b(\xi_3)^2 - \beta y_1 \xi_1 \xi_3$$

Using matrices, one has:

$$-\dot{W} = [\xi_1, \xi_2, \xi_3] \begin{bmatrix} \sigma + 2k & -\frac{1}{2}(\sigma - \frac{1}{4}\beta r - \beta z_1) & -\frac{1}{2}\beta y_1 \\ -\frac{1}{2}(\sigma - \frac{1}{4}\beta r - \beta z_1) & \beta & 0 \\ -\frac{1}{2}\beta y_1 & 0 & \beta b \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}.$$

For  $\rho := 0.1$ ,

$$-\dot{W} - \rho W = [\xi_1, \xi_2, \xi_3] \begin{bmatrix} \sigma + 2k - \frac{\rho}{2} & -\frac{1}{2}\left(\sigma - \frac{1}{4}\beta r - \beta z_1\right) & -\frac{1}{2}\beta y_1 \\ -\frac{1}{2}\left(\sigma - \frac{1}{4}\beta r - \beta z_1\right) & \beta\left(1 - \frac{\rho}{2}\right) & 0 \\ -\frac{1}{2}\beta y_1 & 0 & \beta\left(b - \frac{\rho}{2}\right) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}.$$

Using Sylvester's criteria, and recalling the already proved result that the solution stays inside the bounded set  $A_R$ , it is possible to conclude that  $-\dot{W} - \rho W$  is positive defined if and only if:

1.  $\sigma + 2k - \frac{\rho}{2} > 0$
2.  $(\sigma + 2k - \frac{\rho}{2})\beta(1 - \frac{\rho}{2}) - \frac{1}{4}(\sigma - \frac{1}{4}\beta r - \beta z_1)^2 > 0$
3.  $(\sigma + 2k - \frac{\rho}{2})\beta(1 - \frac{\rho}{2})\beta(b - \frac{\rho}{2}) - \frac{1}{4}\beta^3 y_1^2(1 - \frac{\rho}{2}) - \frac{1}{4}\beta(b - \frac{\rho}{2})(\sigma - \frac{1}{4}\beta r - \beta z_1)^2 > 0.$

It is possible to show that if the third inequality is satisfied, then all the inequalities are satisfied. Therefore, it is enough to study only the last inequality.

Dividing the inequality 3 by  $\beta(b - \frac{\rho}{2})$ , we obtain

$$\left(\sigma + 2k - \frac{\rho}{2}\right) \beta \left(1 - \frac{\rho}{2}\right) > \frac{1}{4} \beta^2 y_1^2 \frac{\left(1 - \frac{\rho}{2}\right)}{\left(b - \frac{\rho}{2}\right)} + \frac{1}{4} \left(\sigma - \frac{1}{4} \beta r - \beta z_1\right)^2 > 0$$

and so,

$$\left(\sigma + 2k - \frac{\rho}{2}\right) > \frac{\beta y_1^2}{4 \left(b - \frac{\rho}{2}\right)} + \frac{\left(\sigma - \frac{1}{4} \beta r - \beta z_1\right)^2}{4\beta \left(1 - \frac{\rho}{2}\right)}.$$

Thus,

$$2k > \frac{\beta y_1^2}{4 \left(b - \frac{\rho}{2}\right)} + \frac{\left(\sigma - \frac{1}{4} \beta r - \beta z_1\right)^2}{4\beta \left(1 - \frac{\rho}{2}\right)} - \sigma + \frac{\rho}{2}.$$

First of all from the attractor estimation, one has:

$$y_1^2 \leq 4424.3$$

$$z_1^2 \leq 4424.3 \Rightarrow -67 \leq z_1 \leq 67.$$

Recalling that  $\beta = 0.1$  and  $\rho = 0.1$  one obtains:

$$2k > \frac{y_1^2}{40 \left(b_m - \frac{\rho}{2}\right)} + 10 \frac{\left(\sigma_M - \frac{1}{40} r_m - \frac{1}{10} z_1\right)^2}{4 \left(1 - \frac{\rho}{2}\right)} - \sigma_M + \frac{\rho}{2}$$

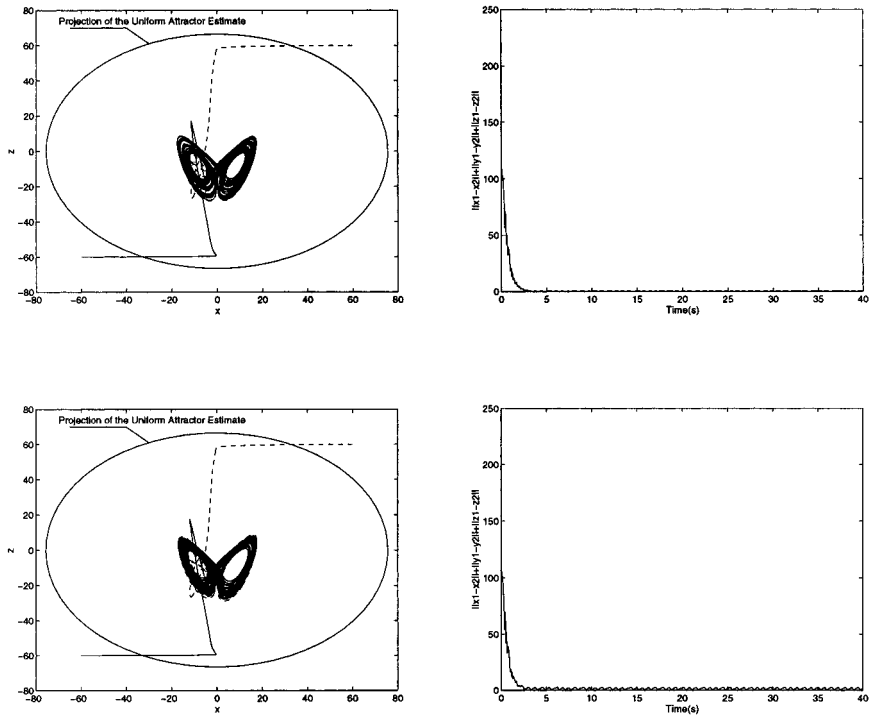
$$2k > 753.7 > \frac{y_1^2}{40 \left(b_m - \frac{\rho}{2}\right)} + 10 \frac{\left(\sigma_M - \frac{1}{40} r_m - \frac{1}{10} z_1\right)^2}{4 \left(1 - \frac{\rho}{2}\right)} - \sigma_M + \frac{\rho}{2}.$$



From  $\dot{W} \leq -\rho W$ , the decay of the evolution operator of (3.21) is obtained.

Therefore the system (3.17) synchronizes for  $k > 377$ .

Figures 7a and 7b show respectively the projection of the orbits of both systems on the plane  $x$ - $z$  and the norm of the difference between the system solutions when the system 2 has an error of  $+1\%$  in the parameters. Figures 7c and 7d are similar to Figs. 7a and 7b but the error in the parameters of the second system is equal to  $+5\%$ . In both situations the systems synchronize as expected from the previous calculations.



**FIG. 7.** Lorenz attractor. (a)  $(x_1(0), y_1(0), z_1(0)) = (-60, 0, -60)$ ,  $(x_2(0), y_2(0), z_2(0)) = (60, 0, 60)$ , for  $\sigma_1 = 10$ ,  $\sigma_2 = 10.1$ ,  $r_1 = 28$ ,  $r_2 = 28.28$ ,  $b_1 = 8/3$ ,  $b_2 = 2.69333\dots$  and  $k = 400$ . (b)  $(x_1(0), y_1(0), z_1(0)) = (-60, 0, -60)$ ,  $(x_2(0), y_2(0), z_2(0)) = (60, 0, 60)$ , for  $\sigma_1 = 10$ ,  $\sigma_2 = 10.1$ ,  $r_1 = 28$ ,  $r_2 = 28.28$ ,  $b_1 = 8/3$ ,  $b_2 = 2.69333\dots$  and  $k = 400$ . (c)  $(x_1(0), y_1(0), z_1(0)) = (-60, 0, -60)$ ,  $(x_2(0), y_2(0), z_2(0)) = (60, 0, 60)$ , for  $\sigma_1 = 10$ ,  $\sigma_2 = 10.5$ ,  $r_1 = 28$ ,  $r_2 = 29.4$ ,  $b_1 = 8/3$ ,  $b_2 = 2.8$  and  $k = 400$ . (d)  $(x_1(0), y_1(0), z_1(0)) = (-60, 0, -60)$ ,  $(x_2(0), y_2(0), z_2(0)) = (60, 0, 60)$ , for  $\sigma_1 = 10$ ,  $\sigma_2 = 10.5$ ,  $r_1 = 28$ ,  $r_2 = 29.4$ ,  $b_1 = 8/3$ ,  $b_2 = 2.8$  and  $k = 400$ .

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